Exact propagator and eigenfunctions for multistable models with arbitrarily prescribed N lowest eigenvalues

1988 J. Phys. A: Math. Gen. 21 2337

(http://iopscience.iop.org/0305-4470/21/10/013)

J. Phys. A: Math. Gen. 21 (1988) 2337-2350. Printed in the UK

Exact propagator and eigenfunctions for multistable models with arbitrarily prescribed N lowest eigenvalues

H R Jauslin

Departement de Physique Théorique, Université de Genève, CH-1211 Genève 4, Switzerland

Received 2 December 1987

Abstract. We present a method to construct potentials for Schrödinger equations with some prescribed features, for which all eigenfunctions and the time-dependent propagator can be explicitly calculated. The prescribed features can be formulated by choosing arbitrarily the N lowest eigenvalues. Alternatively one can prescribe some qualitative behaviour for the potential, like the number and relative depths of wells and barriers. The results can also be applied to the construction of Fokker-Planck models with prescribed properties and explicitly calculable transition probability density. The method is based on ideas of supersymmetric quantum mechanics and the theory of solitons, that can be traced back to the work of Darboux and Crum.

1. Introduction

Many properties of quantum systems, such as those at low temperatures, depend essentially only on the shape of the potential at low energies, or on the lowest eigenvalues and eigenfunctions. In general it is not possible, even in one-dimensional models, to determine the propagator and eigenfunctions for a given problem. It is thus of interest to be able to construct potentials that have a prescribed set of N lowest eigenvalues, for which these quantities can be calculated exactly. The problem can also be formulated in terms of the construction of a potential having some prescribed qualitative features, such as the number and depth of wells, or even some approximate quantitative behaviour at low energies.

The present approach starts with a potential V_0 for which the corresponding propagator and eigenfunctions are explicitly known. V_0 is then modified in such a way as to add N arbitrary eigenvalues at the bottom of the spectrum. This step involves the knowledge of the solutions of the original Schrödinger equation at energies that are smaller than the bottom of the spectrum. The propagator and eigenfunctions for the new potential V_N can be expressed in terms of the original ones, involving only derivatives and integrals. Figure 1 shows some examples of potentials V_N constructed with this method starting from $V_0 = 0$.

In a closely related context, the method allows one to construct multistable Fokker-Planck models, for which the time-dependent transition probabilities can be calculated explicitly.

The basic ideas of this method can be traced back to the work of Darboux [1] and Crum [2]. These ideas have found applications in supersymmetric quantum mechanics [3] and in the study of solitons and inverse scattering theory [4, 5]. Some special cases of this method have been treated in the literature: eigenfunctions have been calculated

0305-4470/88/102337+14$02.50

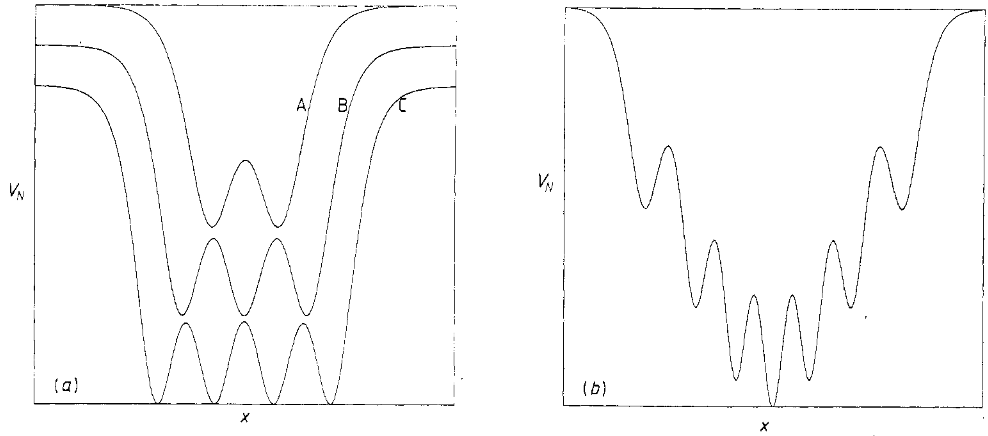

Figure 1. (*a*) Potentials V_N constructed by adding to $V_0=0$ the following eigenvalues λ_i: (A) −1, −1.66; (B) −1, −1.66, −2.637; (C) −1, −1.66, −2.766, −3.269. (The curves are shifted in the V_N direction for clarity.) (*b*) Potential V_N constructed by adding to $V_0=0$ the eigenvalues $\lambda_i=-\mu_i^2$, for $\mu_i=1$, 1.2, 1.4, 1.6, 1.8, 2.0, 2.2.

in [6–9], and time-dependent solutions and propagators were obtained in [10, 11]. In [12, 13] expressions for transition probabilities in bistable Fokker–Planck models were obtained for particular initial conditions.

2. Darboux–Crum transformation

We consider a Schrödinger operator (in the Hilbert space $L_2(R, \mathrm{d}x)$)

$$H_- = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_-(x) \tag{2.1}$$

that has a ground state φ_0 with energy E_0, which we write as

$$\varphi_0 = \mathrm{e}^{-W}. \tag{2.2}$$

Then H_- admits the following representation (with primes denoting derivatives)

$$(H_- - E_0) = A^\dagger A \tag{2.3}$$

with

$$\begin{aligned} A &= \frac{\mathrm{d}}{\mathrm{d}x} + W' \equiv \varphi_0 \frac{\mathrm{d}}{\mathrm{d}x} \varphi_0^{-1} \\ A^\dagger &= -\frac{\mathrm{d}}{\mathrm{d}x} + W' \equiv -\varphi_0^{-1} \frac{\mathrm{d}}{\mathrm{d}x} \varphi_0. \end{aligned} \tag{2.4}$$

We define a new operator

$$H_+ \doteq AA^\dagger + E_0 = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_+ \tag{2.5}$$

where

$$V_+ = V_- + 2W'' \equiv -V_- + 2E_0 + 2W'^2. \tag{2.6}$$

Remark that $V_\pm$ can also be written as

$$V_- = W'^2 - W'' + E_0 \qquad V_+ = W'^2 + W'' + E_0. \tag{2.7}$$

H_+ is called the supersymmetric partner of H_-. Their spectral properties are related as follows.

(1) The spectrum of H_+ is equal to that of H_- but without the eigenvalue E_0.

(2) If $f(x)$ satisfies the equation

$$H_- f = \Gamma f \tag{2.8}$$

where Γ can be

$$\text{either} \quad \text{a real number } \lambda \quad \text{or} \quad \text{the operator } \sigma\frac{\mathrm{d}}{\mathrm{d}t} \quad \text{with } \sigma = \begin{cases} \mathrm{i} \\ -1 \end{cases} \tag{2.9}$$

then $g \doteq Af$ satisfies

$$H_+ g = \Gamma g. \tag{2.10}$$

Conversely, if g satisfies (2.10) then $h \doteq A^\dagger g$ satisfies (2.8).

The two values of $\sigma = \mathrm{i}, -1$ correspond to the Schrödinger and the diffusion (~Fokker–Planck) equations respectively.

Property (1) is proven in [4]. The proof can be sketched as follows. First one shows that E_0 is not an eigenvalue of H_+: by (2.4) the only candidate for eigenfunction is φ_0^{-1}, which is not normalisable. Then one considers the polar decomposition

$$A = U|A| \qquad A^\dagger = |A|U^\dagger \tag{2.11}$$

which gives

$$H_- - E_0 = A^\dagger A = |A|^2 \tag{2.12}$$

$$H_+ - E_0 = AA^\dagger = U|A|^2U^\dagger = U(H_- - E_0)U^\dagger \tag{2.13}$$

where $|A| \doteq (A^\dagger A)^{1/2}$, and U is isometric ($U^\dagger = U^{-1}$) and can be written as

$$U = A(H_- - E_0)^{-1/2}. \tag{2.14}$$

As (2.14) indicates, U is defined only in $K_0^\perp$, the orthogonal complement of the eigenspace of H_- corresponding to E_0. Moreover, U maps $K_0^\perp$ onto the whole space L_2:

$$\begin{aligned} U &: K_0^\perp \to L_2 \\ U^\dagger &: L_2 \to K_0^\perp. \end{aligned} \tag{2.15}$$

Thus, (2.13) states the unitary equivalence between H_+ and the restriction of H_- to the subspace $K_0^\perp$, which implies property (1).

Property (2) is verified by insertion.

This allows us to relate the normalised eigenfunctions $\varphi_{(+),n}$ and $\varphi_{(-),n}$ (for $n \geq 1$) corresponding to the common eigenvalue E_n of H_- and H_+:

$$\varphi_{(-),n} = (E_n - E_0)^{-1/2} A^\dagger \varphi_{(+),n} \tag{2.16}$$

$$\varphi_{(+),n} = (E_n - E_0)^{-1/2} A \varphi_{(-),n}. \tag{2.17}$$

Property (2) allows us to express the time-dependent solutions associated with H_- in terms of time-dependent solutions associated with H_+, and vice versa.

3. Adding one eigenvalue

We start with a Schrödinger operator

$$H_0 = -\frac{d^2}{dx^2} + V_0(x) \tag{3.1}$$

with a potential that grows asymptotically at $\pm\infty$ as

$$V_0 \sim c_\pm x^{\eta_\pm} \qquad c_\pm, \eta_\pm \geq 0. \tag{3.2}$$

We also require that the asymptotic behaviour of its first N derivatives is given by differentiating both sides of (3.2). This includes potentials with a purely discrete spectrum as well as, for example, the free particle. We will denote by λ_0 the lowest point of the spectrum.

We will construct a new Hamiltonian H_1 that has the same spectrum as H_0 but with one supplementary eigenvalue $\lambda_1 < \lambda_0$. The idea is to identify $(H_0 - \lambda_1)$ with the $(H_+ - E_0)$ of the last section; the corresponding H_- is then identified as the new H_1. We want to represent H_0 as

$$(H_0 - \lambda_1) = A_1 A_1^\dagger \tag{3.3}$$

with

$$A_1 = \varphi_{(1),0} \frac{d}{dx} (\varphi_{(1),0})^{-1} = \frac{d}{dx} + W_1'. \tag{3.4}$$

Then

$$H_1 \equiv -\frac{d^2}{dx^2} + V_1(x) \doteq A_1^\dagger A_1 + \lambda_1 \tag{3.5}$$

will have the desired properties.

Equation (3.3) together with (3.4) is equivalent to

$$\begin{aligned} V_0 - \lambda_1 &= (W_1')^2 + W_1'' \\ &\equiv \varphi_{(1),0} \frac{d^2}{dx^2} \varphi_{(1),0}^{-1} \end{aligned} \tag{3.6}$$

or further

$$\left(-\frac{d^2}{dx^2} + V_0\right) \varphi_{(1),0}^{-1} = \lambda_1 \varphi_{(1),0}^{-1}. \tag{3.7}$$

Thus, all that is needed in order to add one eigenvalue is to find a positive solution of the original Schrödinger equation (corresponding to the Hamiltonian (3.1)) for a $\lambda_1 < \lambda_0$, such that its inverse is normalisable. As we will see, for each λ_1 there is a one-parameter family of such solutions. The new potential is given by

$$\begin{aligned} V_1 &= V_0 - 2W_1'' \\ &= -V_0 + 2\lambda_1 + 2\left(\varphi_{(1),0} \frac{d}{dx} \varphi_{(1),0}^{-1}\right)^2 \end{aligned} \tag{3.8}$$

4. Asymptotic properties

In this section we state some results on the existence of solutions of (3.7) that satisfy the required conditions, and characterise them completely. More details and sketches of the proofs are given in the appendix.

(1) There are two linearly independent positive solutions of (3.7) that are uniquely defined by the asymptotic conditions

$$h_1(x) \underset{x\to-\infty}{\rightsquigarrow} (V_0-\lambda_1)^{-1/4}\exp\left(\int^x dy(V_0-\lambda_1)^{1/2}\right)\to 0 \tag{4.1}$$

$$h_3(x) \underset{x\to+\infty}{\rightsquigarrow} (V_0-\lambda_1)^{-1/4}\exp\left(-\int^x dy(V_0-\lambda_1)^{1/2}\right)\to 0. \tag{4.2}$$

h_1 and h_3 go exponentially to ∞ in the opposite limits (see appendix, equations (A6) and (A7)).

(2) With these two functions we can construct a one-parameter family of solutions

$$g_0(x;\lambda_1,\alpha_1) \doteq \alpha_1 h_1(x)+h_3(x) \tag{4.3}$$

that, for $\alpha_1>0$, are everywhere positive and whose inverse is square integrable. We can thus use g_0 for the addition of an eigenvalue. This family, together with h_1 and h_3, are all the positive solutions of (3.7) (up to multiplication with a constant), because h_1 and h_3 are linearly independent, and a negative α_1 would lead to negative values for $x\to\infty$.

Remark that if V_0 is symmetric and we set $\alpha_1=1$ then g_0 is the (unique) even solution, whereas $\alpha_1=-1$ gives the odd solution.

(3) The new potential V_1 has the same asymptotic behaviour (3.2) as V_0. This is a consequence of (3.8) and property (k) of the appendix.

5. Adding two eigenvalues

One could now iterate the procedure of § 3 and add eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_N$ successively. This involves the determination of a positive solution of a new Schrödinger equation at each step. Since the potentials become more and more complicated this may look like a hopeless task. However, it turns out that it is sufficient to know the solutions $h_1(x;\lambda)$ and $h_3(x;\lambda)$ of the original H_0 for $\lambda=\lambda_1,\lambda_2,\ldots,\lambda_N$, and that one can construct H_N in one step.

In order to explain the procedure, we consider first the addition of a second eigenvalue λ_2. Let $g_0(x;\lambda_1,\alpha_1)$ be a positive solution for $\lambda=\lambda_1$ of

$$H_0 f=\lambda f \tag{5.1}$$

as defined in (4.3). Define

$$V_1=V_0-2\frac{d^2}{dx^2}\ln g_0. \tag{5.2}$$

We know from § 2 that, if $f_0(x;\lambda)$ is a solution of (5.1) for an arbitrary λ, then a solution of

$$H_1 f=\lambda f \tag{5.3}$$

is given by

$$f_1(x;\lambda) \doteq cA_1^\dagger f_0(x;\lambda) \tag{5.4}$$

$$\equiv -cg_0^{-1}(\lambda_1,\alpha_1)\det\begin{pmatrix} g_0(\lambda_1,\alpha_1) & f_0(\lambda) \\ g_0'(\lambda_1,\alpha_1) & f_0'(\lambda)\end{pmatrix} \tag{5.5}$$

$$\equiv -c\frac{\Omega_2(g_0(\lambda_1,\alpha_1), f_0(\lambda))}{\Omega_1(g_0(\lambda_1,\alpha_1))} \tag{5.6}$$

where c is a constant and

$$A_1^\dagger = -g_0(\lambda_1,\alpha_1)\frac{\mathrm{d}}{\mathrm{d}x}g_0^{-1}(\lambda_1,\alpha_1). \tag{5.7}$$

In (5.6) we have introduced the notation

$$\Omega_n(f_1, f_2, \ldots, f_n) \doteq \det\begin{pmatrix} f_1 & \cdots & f_n \\ f_1' & \cdots & f_n' \\ \vdots & \ddots & \vdots \\ f_1^{(n-1)} & & f_n^{(n-1)}\end{pmatrix} \tag{5.8}$$

and $\Omega_0 \doteq 1$. In fact, all the solutions can be obtained this way. The general positive solution g_1 is given by

$$g_1 = \frac{\Omega_2(g_0(\lambda_1,\alpha_1), g_0(\lambda_2,-\alpha_2))}{\Omega_1(g_0(\lambda_1,\alpha_1))} \tag{5.9}$$

where $\alpha_2 > 0$ is an arbitrary constant. Notice that the sign in $-\alpha_2$ has the effect that

$$\Omega_2 \xrightarrow[x\to\pm\infty]{} +\infty. \tag{5.10}$$

This implies, by the arguments of the appendix, that g_1 is positive and its inverse is normalisable.

Therefore we can construct the new potential with two added eigenvalues as

$$\begin{aligned} V_2 &= V_1 - 2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln g_1 \\ &= V_0 - 2\frac{\mathrm{d}^2}{\mathrm{d}x^2}(\ln g_1 + \ln g_0) \\ &= V_0 - 2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln\frac{\Omega_2(g_0(\lambda_1,\alpha_1), g_0(\lambda_2,-\alpha_2))}{\Omega_1(g_0(\lambda_1,\alpha_1))}\Omega_1(g_0(\lambda_1,\alpha_1)) \\ &= V_0 - 2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln\Omega_2(g_0(\lambda_1,\alpha_1), g_0(\lambda_2,-\alpha_2)). \end{aligned} \tag{5.11}$$

Its ground state and the eigenfunction corresponding to λ_1 are

$$\varphi_{(2),\lambda_2} = c_{2,2}g_1^{-1} \tag{5.12}$$

$$\varphi_{(2),\lambda_1} = c_{2,1}A_2^\dagger g_0^{-1} \tag{5.13}$$

where $c_{2,2}$ and $c_{2,1}$ are normalisation constants. The normalised eigenfunctions for the eigenvalues $E_n > \lambda_1$ are given by

$$\varphi_{(2),n} = (E_n - \lambda_2)^{-1/2} A_2^\dagger \varphi_{(1),n} \tag{5.14}$$

$$= (E_n - \lambda_2)^{-1/2} (E_n - \lambda_1)^{-1/2} A_2^\dagger A_1^\dagger \varphi_{(0),n} \tag{5.15}$$

$$= Z_{2,n} B_2^\dagger \varphi_{(0),n}. \tag{5.16}$$

In the last equality we have introduced the notation

$$B_N^\dagger \doteq A_N^\dagger A_{N-1}^\dagger \ldots A_1^\dagger \tag{5.17}$$

with

$$A_i^\dagger = -g_{i-1} \frac{\mathrm{d}}{\mathrm{d}x} g_{i-1}^{-1} \tag{5.18}$$

and for the normalisation

$$Z_{N,n} \doteq \prod_{i=1}^{N} (E_n - \lambda_i)^{-1/2}. \tag{5.19}$$

Notice that the operator $B_2^\dagger$ acting on a function f can be represented as

$$B_2^\dagger f = \frac{\Omega_3(g_0(\lambda_1, \alpha_1), g_0(\lambda_2, -\alpha_2), f)}{\Omega_2(g_0(\lambda_1, \alpha_1), g_0(\lambda_2, -\alpha_2))}. \tag{5.20}$$

6. Adding N eigenvalues

The results of the previous section are easily extended to the simultaneous addition of N eigenvalues. Given N arbitrary values $\lambda_0 > \lambda_1 > \lambda_2 > \ldots > \lambda_N$ and N positive constants $\alpha_1, \ldots, \alpha_N$, we construct the potential

$$V_N = V_0 - 2\frac{\mathrm{d}^2}{\mathrm{d}x^2} \ln \Omega_N(g_0(\lambda_1, +\alpha_1), g_0(\lambda_2, -\alpha_2), g_0(\lambda_3, +\alpha_3), \ldots, g_0(\lambda_N, (-1)^{N+1}\alpha_N)). \tag{6.1}$$

The corresponding Hamiltonian H_N has the same spectrum as H_0 plus the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_N$. Its ground state is

$$\varphi_{(N),\lambda_N} = c_{N,N} g_{N-1}^{-1} \tag{6.2}$$

where g_{N-1} is the general positive solution of $H_{N-1} g_{N-1} = \lambda_N g_{N-1}$, which is given by (6.6). The eigenfunctions corresponding to $\lambda_1, \lambda_2, \ldots, \lambda_{N-1}$ are

$$\varphi_{(N),\lambda_i} = c_{N,i} A_N^\dagger \ldots A_{i+1}^\dagger g_{i-1}^{-1}. \tag{6.3}$$

The normalised eigenfunctions for the eigenvalues $E_n > \lambda_1$ are given by

$$\varphi_{(N),n} = Z_{N,n} B_N^\dagger \varphi_{(0),n} \tag{6.4}$$

(with the notation introduced in (5.17) and (5.19)).

The operator $B_N^\dagger$ can be represented by

$$B_N^\dagger f = (-1)^N \frac{\Omega_{N+1}(g_0(\lambda_1, \alpha_1), \ldots, g_0(\lambda_N, (-1)^{N+1}\alpha_N), f)}{\Omega_N(g_0(\lambda_1, \alpha_1), \ldots, g_0(\lambda_N, (-1)^{N+1}\alpha_N))}. \tag{6.5}$$

This relation follows immediately from

$$g_{N-1}=\frac{\Omega_N(g_0(\lambda_1,\alpha_1),\ldots,g_0(\lambda_N,(-1)^{N+1}\alpha_N))}{\Omega_{N-1}(g_0(\lambda_1,\alpha_1),\ldots,g_0(\lambda_{N-1},(-1)^N\alpha_{N-1}))} \tag{6.6}$$

which is obtained by induction from the results of § 5. (This is done, for example, in [5, p 176] for the case $V_0=0$, but the algebra is the same in the general case.)

Therewith we obtain (6.1) by

$$\begin{aligned} V_N &= V_{N-1}-2\frac{d^2}{dx^2}\ln\frac{\Omega_N}{\Omega_{N-1}} \\ &= V_{N-2}-2\frac{d^2}{dx^2}\ln\frac{\Omega_N}{\Omega_{N-1}}\frac{\Omega_{N-1}}{\Omega_{N-2}} \\ &\;\;\vdots \\ &= V_0-2\frac{d^2}{dx^2}\ln\Omega_N. \end{aligned} \tag{6.7}$$

The rest is an immediate consequence of the results of the previous sections.

7. Propagators

In this section we show that the time-dependent propagators G_N and G_{N-1} corresponding to the Hamiltonians H_N and H_{N-1} are related by the two following equivalent equations:

$$G_N(x,x_0;t)=-g_{N-1}^{-1}(x_0)A_N^\dagger(x)\int_{x_0}^{\infty}dz\,G_{N-1}(x,z,t)g_{N-1}(z) \tag{7.1}$$

$$\begin{aligned} G_N(x,x_0;t)&=\theta(t)\exp(\sigma^{-1}\lambda_N t)\varphi_{(N),\lambda_N}(x)\varphi_{(N),\lambda_N}(x_0) \\ &\quad-\sigma^{-1}\theta(t)A_N^\dagger(x)A_N^\dagger(x_0)\int_t^{\infty}ds\,G_{N-1}(x,x_0,s)\exp[\sigma^{-1}\lambda_N(t-s)]. \end{aligned} \tag{7.2}$$

By iteration we can express G_N in terms of the known G_0. The result is given in (7.21) and (7.22). The validity of (7.1) and (7.2) can be checked directly by insertion. They were constructed as follows.

(1) For the construction of (7.1) we forget the Hilbert space structure and just consider the differential equations. In § 2 we saw that the time-dependent solutions of

$$\sigma\frac{d}{dt}\psi_N(x,t)=H_N\psi_N(x,t) \tag{7.3}$$

can be expressed in terms of solutions of

$$\sigma\frac{d}{dt}\psi_{N-1}(x,t)=H_{N-1}\psi_{N-1}(x,t) \tag{7.4}$$

by

$$\psi_N(x,t)=A_N^\dagger\psi_{N-1}(x,t). \tag{7.5}$$

Notice that in (7.5) we do not ask that the ψ are in the Hilbert space.

Thus, in order to relate the propagators G_N and G_{N-1}, we need to determine the initial condition $\psi_{N-1}(x, 0)$ that gives

$$A_N^\dagger \psi_{N-1}(x, 0) = \psi_N(x, 0) = \delta(x - x_0). \tag{7.6}$$

The inversion of (7.6) gives

$$\begin{aligned}\psi_{N-1}(x, 0) &= -g_{N-1}(x) \int_{-\infty}^{x} dz\, g_{N-1}^{-1}(z)\delta(z - x_0) \\ &= -g_{N-1}(x) g_{N-1}^{-1}(x_0)(1 - \theta(x - x_0)).\end{aligned} \tag{7.7}$$

($\psi_{N-1}(x, 0)$ is clearly not in L_2.) The propagator is given by

$$\begin{aligned}G_N(x, x_0, t) &= \theta(t) A_N^\dagger(x) \psi_{N-1}(x, t) \\ &= A_N^\dagger(x) \int_{-\infty}^{\infty} dz\, G_{N-1}(x, z, t) \psi_{N-1}(z, 0).\end{aligned} \tag{7.8}$$

Inserting (7.7) into (7.8) we obtain (7.1).

(2) To obtain the second expression (7.2) we work in the Hilbert space. There the general solutions of (7.3) and (7.4) are related by

$$\psi_{N-1}(x, t) = A_N \psi_N(x, t) \tag{7.9}$$

$$\psi_N(x, t) = A_N^\dagger \psi_{N-1}(x, t) + c\varphi_{(N),\lambda_N}(x) \exp(\sigma^{-1}\lambda_N s) \tag{7.10}$$

where c is a constant which is given, e.g. for the initial condition $\psi_N(x, 0) = \delta(x - x_0)$, by

$$c = \langle \psi_N(x, 0), \varphi_{(N),\lambda_N}(x) \rangle = \varphi_{(N),\lambda_N}(x_0). \tag{7.11}$$

The last term in (7.10) must be added because all the functions of the form $A_N^\dagger \phi$ are orthogonal to $\varphi_{(N),\lambda_N}$:

$$\langle A_N^\dagger \phi, \varphi_{(N),\lambda_N} \rangle = \langle \phi, A_N \varphi_{(N),\lambda_N} \rangle = 0. \tag{7.12}$$

Every function ψ of L_2 (i.e. every admissible initial condition) can be expressed as $\psi = A_N \phi$, with the appropriate choice of ϕ in L_2. However, the functions of the form $\psi = A_N^\dagger \phi$ give only all the functions that are orthogonal to $\varphi_{(N),\lambda_N}$. Notice that in (7.5) the necessity of this supplementary term is avoided by allowing functions that are not in the Hilbert space.

Now, to obtain the second expression (7.2), instead of the inversion (7.7) we let A_N act on (7.10) and get

$$(H_{N-1} - \lambda_N) \psi_{N-1}(x, 0) = A_N \psi_N(x, 0). \tag{7.13}$$

The operator $(H_{N-1} - \lambda_N)$ is invertible and we can write

$$\psi_{N-1}(x, 0) = R_{N-1} A_N \psi_N(x, 0) \tag{7.14}$$

where $R_{N-1} \doteq (H_{N-1} - \lambda_N)^{-1}$ is the resolvent, whose integral kernel is

$$\tilde{R}_{N-1}(x, y) = -\sigma^{-1} \int_0^{\infty} ds\, G_{N-1}(x, y, s) \exp(-\sigma^{-1}\lambda_N s). \tag{7.15}$$

$G_N(x, x_0, t)$ is the solution with initial condition $\delta(x - x_0)$. By the preceding arguments it can be written as

$$G_N(x, x_0, t) = \theta(t)\varphi_{(N),\lambda_N}(x)\varphi_{(N),\lambda_N}(x_0) \exp(\sigma^{-1}\lambda_N s) + \theta(t) A_N^\dagger(x) \hat{\psi}_{N-1}(x, x_0, t) \tag{7.16}$$

where $\hat{\psi}_{N-1}(x, x_0, t)$ is the solution of (7.4) with initial condition (according to (7.14)) $\hat{\psi}_{N-1}(x, x_0, 0) = R_{N-1}A_N(x)\delta(x-x_0)$. It is given by

$$\begin{aligned}\hat{\psi}_{N-1}(x, x_0, t) &= \exp(\sigma^{-1}H_{N-1}t)R_{N-1}A_N\delta(x-x_0)\\ &= R_{N-1}\int_{-\infty}^{\infty} \mathrm{d}y\, G_{N-1}(x, y, t)A_N\delta(y-x_0)\\ &= R_{N-1}\int_{-\infty}^{\infty} \mathrm{d}y\, \delta(y-x_0)A_N^{\dagger}(y)G_{N-1}(x, y, t)\\ &= A_N^{\dagger}(x_0)\int_{-\infty}^{\infty} \mathrm{d}z\, \tilde{R}_{N-1}(x, z)G_{N-1}(z, x_0, t)\\ &= -\sigma^{-1}A_N^{\dagger}(x_0)\int_t^{\infty} \mathrm{d}s\, G_{N-1}(x, x_0, s)\exp[\sigma^{-1}\lambda_N(t-s)].\end{aligned} \tag{7.17}$$

For the last equality we have used (7.15) and the identity

$$G_{N-1}(x, x_0, s) = \int_{-\infty}^{\infty} \mathrm{d}z\, G_{N-1}(x, z, s-t)G_{N-1}(z, x_0, t). \tag{7.18}$$

This, together with (7.16), gives the result (7.2).

We remark that this result can also be obtained in a simple way in the case of a purely discrete spectrum, using the representation

$$G(x, x_0, t) = \theta(t)\sum_{n=0}^{\infty} \varphi_n(x)\varphi_n(x_0)\exp(\sigma^{-1}E_n t) \tag{7.19}$$

and applying the transformation (2.16) to $\varphi_n(x)$ and $\varphi_n(x_0)$ and the identity

$$\begin{aligned}&-\sigma^{-1}\int_t^{\infty} \mathrm{d}s\, G_{N-1}(x, x_0, s)\exp[\sigma^{-1}\lambda(t-s)]\\ &\qquad = \theta(t)\sum_{n=0}^{\infty} \varphi_n(x)\varphi_n(x_0)(E_n-\lambda)^{-1}\exp(\sigma^{-1}E_n t).\end{aligned} \tag{7.20}$$

(3) We remark finally that after performing the iteration of (7.1) the result can be written compactly as

$$\begin{aligned}G_N(x, x_0; t) = (-1)^N B_N^{\dagger}(x)&\int_{x_0}^{\infty} \mathrm{d}z_N \int_{z_N}^{\infty} \mathrm{d}z_{N-1} \ldots \int_{z_2}^{\infty} \mathrm{d}z_1\, G_0(x, z_1, t)\\ &\times \prod_{i=1}^{N} (g_{i-1}^{-1}(z_{i+1})g_{i-1}(z_i))\end{aligned} \tag{7.21}$$

with $z_{N+1} \equiv x_0$. The iteration of (7.2) gives (with $s_{N+1} \equiv t$)

$$\begin{aligned}G_N(x, x_0; t) &= \theta(t)\sum_{i=1}^{N} \exp(\sigma^{-1}\lambda_N t)\varphi_{(i),\lambda_i}(x)\varphi_{(i),\lambda_i}(x_0)\\ &\quad + \theta(t)(-\sigma)^{-N}B_N^{\dagger}(x)B_N^{\dagger}(x_0)\int_t^{\infty} \mathrm{d}s_N \int_{s_N}^{\infty} \mathrm{d}s_{N-1} \ldots\\ &\quad \times \int_{s_2}^{\infty} \mathrm{d}s_1\, G_0(x, x_0, s_1)\exp\left(\sigma^{-1}\sum_{i=1}^{N} (s_{i+1}-s_i)\lambda_i\right).\end{aligned} \tag{7.22}$$

8. Conclusion

We have thus a method to construct families of exactly solvable models depending on $2N$ free parameters. One can use this freedom to construct potentials that have certain qualitative features that mimic models for particular physical problems.

The asymptotic behaviour is the same within a family. Thus the modifications occur in relatively localised regions. For instance, one observes that, if two eigenvalues are put close to each other, the potential develops a barrier between two wells, as one may expect from the theory of tunnel splitting.

The case $V_0=0$ is related to soliton solutions of the Korteweg-de Vries equation [14]. Since the number of solitons is equal to the number of eigenvalues, an appropriate choice of the parameters can give one potential well per eigenvalue.

The choice of V_0 is limited in practice by the need to know explicitly the general solution of the corresponding stationary Schrödinger equation as well as the propagator. Some examples are $V_0=0$, $V_0=ax^2$ and $V_0=ax^2+bx^{-2}$ (see [15, 16]).

We remark finally that this method allows us also to construct Fokker-Planck models with prescribed properties for which the transition probability density and the eigenfunctions can be explicitly calculated. The transition probability density corresponding to the Fokker-Planck equation

$$\frac{\mathrm{d}}{\mathrm{d}t}P=-\frac{\mathrm{d}}{\mathrm{d}x}(KP)+\frac{\mathrm{d}^2}{\mathrm{d}x^2}P \tag{8.1}$$

with

$$K(x)=2\frac{\mathrm{d}}{\mathrm{d}x}\ln\varphi_{(N),\lambda_N}(x) \tag{8.2}$$

is given by

$$P(x,t|x_0)=\frac{\varphi_{(N),\lambda_N}(x)}{\varphi_{(N),\lambda_N}(x_0)}G_N(x,x_0,t)\exp(\lambda_N t). \tag{8.3}$$

For example we consider the potential V_1 constructed by adding one eigenvalue $\lambda_1=-\mu^2$ to the free particle ($V_0=0$). The general positive solution of (3.7) is

$$g_0(x;-\mu^2,\alpha)=\mathrm{ch}(\mu x;\alpha)\doteq\tfrac{1}{2}(\alpha\,\mathrm{e}^{\mu x}+\mathrm{e}^{-\mu x}). \tag{8.4}$$

The new potential is (by (3.8)):

$$V_1=\alpha\mu/\mathrm{ch}(\mu x;\alpha) \tag{8.5}$$

and the normalised ground state

$$\varphi_{1,\lambda_1}(x)=(2\mu\alpha)^{1/2}/\mathrm{ch}(\mu x;\alpha). \tag{8.6}$$

The propagator can be easily calculated using (7.1):

$$G(x,x_0;t)=\frac{1}{(4\pi t)^{1/2}}\exp\left(-\frac{(x-x_0)^2}{4t}\right)+\frac{\mu\exp(\mu^2 t)}{4\,\mathrm{ch}(\mu x_0;\alpha)\,\mathrm{ch}(\mu x;\alpha)}\times\left[\mathrm{erf}\left(\frac{2\mu t+(x-x_0)}{\sqrt{4t}}\right)+\mathrm{erf}\left(\frac{2\mu t-(x-x_0)}{\sqrt{4t}}\right)\right] \tag{8.7}$$

where erf denotes the error function.

We can now consider the Fokker–Planck equation (8.1) with drift

$$K=-2\mu\ \mathrm{th}(\mu x;\ \alpha)\doteq-2\mu\frac{\alpha\ \mathrm{e}^{\mu x}-\mathrm{e}^{-\mu x}}{\alpha\ \mathrm{e}^{\mu x}+\mathrm{e}^{-\mu x}}. \tag{8.8}$$

Its stationary state is

$$P_s=\varphi^2_{1,\lambda_1}(x)=\frac{2\mu\alpha}{\mathrm{ch}^2(\mu x;\ \alpha)} \tag{8.9}$$

and the transition probability density is given by

$$\begin{aligned}P(x,t|x_0)=&\frac{\mathrm{ch}(\mu x_0;\ \alpha)}{\mathrm{ch}(\mu x;\ \alpha)}\frac{1}{(4\pi t)^{1/2}}\exp\left(-\frac{(x-x_0)^2}{4t}\right)\exp(-\mu^2 t)\\&+\frac{\mu\alpha}{4\ \mathrm{ch}^2(\mu x;\ \alpha)}\left[\mathrm{erf}\left(\frac{2\mu t+(x-x_0)}{\sqrt{4t}}\right)+\mathrm{erf}\left(\frac{2\mu t-(x-x_0)}{\sqrt{4t}}\right)\right].\end{aligned} \tag{8.10}$$

Acknowledgments

I would like to thank L S Schulman, F Monti and M-O Hongler for very useful discussions. This work has been partially supported by the Fonds National Suisse de la Recherche Scientifique.

Appendix. Asymptotic properties

In this appendix we discuss some asymptotic properties of the solutions of the Schrödinger equation

$$-g''+Vg=\lambda g \qquad \lambda<\lambda_0 \tag{A1}$$

where V has the asymptotic behaviour (3.2) and λ is smaller than the lowest point of the spectrum λ_0. We use some classical results that can be found, e.g., in [17-20].

(a) $\lambda<\lambda_0$ implies that g has at most one zero and it is simple. The proof can be found, e.g., in [17, p 208].

(b1) There are solutions h_1, h_2 that have the following asymptotic behaviour at $-\infty$:

$$h_1(x) \underset{x\to-\infty}{\rightsquigarrow} (V-\lambda)^{-1/4}\exp\left(\int^x \mathrm{d}y(V-\lambda)^{1/2}\right)\to 0 \tag{A2}$$

$$h_2(x) \underset{x\to-\infty}{\rightsquigarrow} (V-\lambda)^{-1/4}\exp\left(-\int^x \mathrm{d}y(V-\lambda)^{1/2}\right)\to \infty. \tag{A3}$$

(b2) There are solutions h_3, h_4 that have the following asymptotic behaviour at $+\infty$:

$$h_3(x) \underset{x\to+\infty}{\rightsquigarrow} (V-\lambda)^{-1/4}\exp\left(-\int^x \mathrm{d}y(V-\lambda)^{1/2}\right)\to 0 \tag{A4}$$

$$h_4(x) \underset{x\to+\infty}{\rightsquigarrow} (V-\lambda)^{-1/4}\exp\left(\int^x \mathrm{d}y(V-\lambda)^{1/2}\right)\to \infty. \tag{A5}$$

This is proven, e.g., in [19, pp 190-202].

(c) h_1 and h_3 are uniquely defined by (A2) and (A4). (This is not the case for h_2 and h_4.)

Proof. If there were another linearly independent solution with the same asymptotic behaviour, then all solutions would tend to zero in that limit, which is in contradiction with (A3) and (A5).

(d) h_1 is square integrable in an interval $(-\infty, x_1)$ for some small enough x_1 and h_3 is square integrable in an interval (x_3, ∞) for some large enough x_3. This is an immediate consequence of the exponential decay.

(e) The following are pairs of linearly independent solutions: (h_1, h_2), (h_3, h_4), and (h_1, h_3).

Proof. For the first two pairs it is evident from the asymptotics. If h_1 and h_3 were linearly dependent, they would be square integrable at both limits, and thus they would be eigenfunctions, which contradicts $\lambda < \lambda_0$.

(f) The following behaviour follows from (b)-(e):

$$h_1(x) \underset{x\to+\infty}{\rightsquigarrow} c_1(V-\lambda)^{-1/4}\exp\left(\int^x dy(V-\lambda)^{1/2}\right)\to\infty \qquad (A6)$$

$$h_3(x) \underset{x\to-\infty}{\rightsquigarrow} c_3(V-\lambda)^{-1/4}\exp\left(-\int^x dy(V-\lambda)^{1/2}\right)\to\infty \qquad (A7)$$

where c_1 and c_3 are constants.

(g) h_1 and h_3 are everywhere positive.

Proof. Consider h_1. For $x\to-\infty$ it is positive by definition. Since we know that it can have at most one zero (counting multiplicity), we must exclude two situations: (i) h_1 becomes negative and approaches zero at $+\infty$; (ii) $h_1\to-\infty$ for $x\to+\infty$ (plus the analogue cases for h_3). (i) can be excluded because otherwise h_1 would be an eigenfunction. To exclude (ii) we first note that h_1 and h_3 cannot both have zeros, since otherwise one could construct a linear combination ah_1+bh_3 which, for $a, b>0$ and b small enough, would have two zeros. Assume then, e.g., that h_1 has a zero and h_3 none. The choice of a negative a and a small enough $b>0$ would lead again to a solution with two zeros. The analogue argument for the reversed situation completes the proof.

(h) Therefore, the one-parameter family of solutions

$$g(x, \lambda, \alpha) \doteq \alpha h_1(x) + h_3(x) \qquad \alpha > 0 \qquad (A8)$$

is also everywhere positive. As we mentioned in § 4, these are all the positive solutions.

(i) Properties (b) and (f) imply that the g^{-1} are normalisable. This gives us the one-parameter family of positive solutions that we can use to add an eigenvalue.

(j) For $|x|$ large enough, h_1 and h_3 as well as their first derivatives are monotonic functions.

Proof. Equation (A1), together with $\lambda < \lambda_0$ and the positivity of h_1 and h_3, imply that there is a y such that for all $|x| > |y|$ one has $h_1'' > 0$ and $h_3'' > 0$, which implies the monotonicity of the first derivatives. Finally we show that asymptotically they have a constant sign, which implies the monotonicity of h_1 and h_3 for large $|x|$. Consider h_1. There is an $x_1 < -|y|$, at which $h_1' > 0$. It then decreases toward zero as $x \to -\infty$ without ever becoming negative, since then it would stay negative and h_1 would diverge. On the other side there is an $x_2 > |y|$, at which $h_1' > 0$. Then it stays positive for all $x > x_2$. Analogous arguments apply to h_3'.

(k) This allows us to obtain the asymptotics of the derivatives from the asymptotics of the functions [20, pp 49–53]:

$$h_1'(x) \underset{x\to-\infty}{\rightsquigarrow} [-\tfrac{1}{4}V'(V-\lambda)^{-5/4}+(V-\lambda)^{1/4}]\exp\left(\int^x \mathrm{d}y(V-\lambda)^{1/2}\right) \to 0 \qquad \text{(A9)}$$

$$h_3'(x) \underset{x\to+\infty}{\rightsquigarrow} [-\tfrac{1}{4}V'(V-\lambda)^{-5/4}-(V-\lambda)^{1/4}]\exp\left(-\int^x \mathrm{d}y(V-\lambda)^{1/2}\right) \to 0. \qquad \text{(A10)}$$

References

[1] Darboux G 1882 *C.R. Acad. Sci., Paris* **94** 1456; 1894 *Théorie des surfaces* vol 2 (Paris: Gauthier-Villars)
[2] Crum M M 1955 *Q. J. Math.* **6** 121
[3] Witten E 1981 *Nucl. Phys.* B **188** 513
[4] Deift P A 1978 *Duke Math. J.* **45** 267
[5] Deift P and Trubowitz E 1979 *Commun. Pure Appl. Math.* **29** 121
[6] Zheng W M 1984 *J. Math. Phys.* **25** 88
[7] Razavy M 1980 *Am. J. Phys.* **48** 285
[8] Mielnik B 1984 *J. Math. Phys.* **25** 3387
[9] Sukumar C V 1985 *J. Phys. A: Math. Gen.* **18** 2917, 2937; 1986 *J. Phys. A: Math. Gen.* **19** 2297; 1987 *J. Phys. A: Math. Gen.* **20** 2461
[10] Gaveau B and Schulman L S 1986 *J. Phys. A: Math. Gen.* **19** 1833
[11] Englefield M J 1987 *J. Phys. A: Math. Gen.* **20** 593
[12] Hongler M-O and Zheng W M 1982 *J. Stat. Phys.* **29** 317
[13] Hongler M-O and Zheng W M 1983 *J. Math. Phys.* **24** 336
[14] Lamb G L Jr 1980 *Elements of Soliton Theory* (New York: Wiley)
[15] Infeld L and Hull T E 1951 *Rev. Mod. Phys.* **23** 21
[16] Schulman L S 1981 *Techniques and Applications of Path Integration* (New York: Wiley)
[17] Coddington E A and Levinson N 1984 *Theory of Ordinary Differential Equations* (Malabas, FL: Krieger) 2nd edn
[18] Titchmarsh E C 1962 *Eigenfunction Expansions Associated with Second-order Differential Equations* (Oxford: Oxford University Press)
[19] Olver F W 1974 *Asymptotics and Special Functions* (New York: Academic)
[20] Berg L 1968 *Asymptotische Darstellungen und Entwicklungen* (Berlin: Deutscher Verlag der Wissenschaften)